Old and new in complex dynamical systems

David Shoikhet (Holon Institute of Technology, Israel)

One of the first applicative models of the complex dynamical systems on the unit disk arose more than hundred years ago in investigations of dynamics of stochastic branching processes.

In 1874 F.Galton and H.W. Watson, in treating the problem of the extinction probability of family names formulated a mathematical model in terms of the probability generating function:

$$F(z) = \sum_{k=0}^{\infty} p_k z^k \quad , |z| \leq 1$$

where z is a complex variable, $p_0, p_1, \ldots, p_k$ are nonnegative numbers (probabilities) such that $\sum_{k=0}^{\infty} p_k = 1$, and its iterations:

$$F^{(0)}(z) = z, \quad F^{(n+1)}(z) = F^{(n)}(F(z)).$$

The first complete and correct determination of the extinction probability for the Galton - Watson process as the limit points of the iteration sequence was given by J.F. Steffensen in 1930. Since that the interest in this model has increased because of connections with chemical and nuclear chain reactions, the study of the multiplication of electrons in the electron multiplier, the theory of cosmic radiation and many other biological and physical problems.

Note that if the original Galton - Watson process related to discrete - time branching process (i.e. it is described by iteration process of a single probability generating function) the further development involved also the consideration of continuous time branching processes based on one-parameter semigroups of analytical self-mappings of the unit disk. One of the problems in analysis is, given a function $F(z)$, to find a function $F(z,t)$, with $F(z,1) = F(z)$ satisfying the semigroup property

$$F(z, t+s) = F(F(t,z), s), \quad t, s \geq 0$$

where z is a complex variable. Since this formula expresses the characteristic property of iteration when t and s are integers we may consider $F(z,t)$ as a fractional iterate of F, when t is not an integer.

Koenigs (1884) showed how this problem may be solved, if F is analytic self-mapping on the unit disk with an interior fixed point $z_0 = F(z_0)$, such that $0 < |F'(z_0)| < 1$, by using the convergence of the sequence $\left\{F^{(n)}(z)\right\}$ to z_0, as $n \to \infty$ in a neighborhood of the point z_0.

These and other problems led to the so called Denjoy-Wolff Theory which since 1926 has been developed in many directions in terms of classical dynamics in finite and infinite complex spaces as well as in terms of the hyperbolic geometry.

In this talk we consider also the asymptotic behavior of semigroups generated by holomorphic functions by using infinitesimal versions of the Schwarz-Wolff Lemma and the Julia-Caratheodory Theorem.

These enable us to describe the class of univalent functions which are starlike and spirallike with respect to a boundary point.

In addition, we discuss conditions which ensure the existence of backward flow invariant sets for semigroups of holomorphic self-mappings of a domain D. More precisely, the problem is the following.

> Given a one-parameter semigroup $S = \{F(z,t)\}$ on D, find a simply connected subset Ω in D (if it exists) such that each element of S is an automorphism of Ω, in other words, such that S forms a one-parameter group on Ω.

Further we mention a deep relationship between complex dynamics and the theory of composition operators on Hardy and Bergman spaces. In particular, we consider the eigen-value problem of composition operators defined by Schroeder's functional equation for a semigroup with a boundary Wolff's point.