Contractive realization of rational functions on multiply connected domains

Radomił Baran (Jagiellonian University, Kraków, Poland)

A classical result due to Arov [1] asserts that a rational matrix function $F(z)$ of size $k \times l$, analytic and contractive on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, admits a contractive finite-dimensional realization. Specifically, there exists a contractive block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(d+k)\times(d+l)}$$

such that

$$F(z) = D + zC(I - zA)^{-1}B, \quad z \in \mathbb{D}.$$

A generalization of this result, developed in [2], provides sufficient conditions for the existence of a contractive realization of the form

$$F(z) = D + C\mathbf{P}(z)(I - A\mathbf{P}(z))^{-1}B,$$

where $\mathbf{P}$ is a matrix-valued polynomial and the domain is defined by the inequality $I - \mathbf{P}(z)^*\mathbf{P}(z) \geq 0$. A key assumption involves boundedness in the so-called Agler norm:

$$\|F\|_{\mathcal{A},\mathbf{P}} := \sup\{\|F(\mathbf{T})\|\},$$

where the supremum is taken over commuting tuples $\mathbf{T}$ of bounded operators on a Hilbert space such that $\|\mathbf{P}(\mathbf{T})\| < 1$. This framework applies to rational functions regular on polynomially convex domains.

In [3], we extend this theory beyond the polynomially convex case. In particular, we establish a contractive realization theorem for rational matrix functions without poles in the annulus and, more generally, in multihole domains. Our proof builds on techniques developed in [2] and and incorporates tools from the theory of operator algebras.

This talk is based on joint work with Piotr Pikul, Hugo J. Woerdeman, and Michał Wojtylak.

[1] D. Z. Arov: Passive linear steady-state dynamical systems, *Sibirsk. Mat. Zh.* 20 (1979), no. 2, 211–228, 457.

[2] A. Grinshpan, D. S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. J. Woerdeman: Matrix-valued Hermitian Positivstellensatz, lurking contractions, and contractive determinantal representations of stable polynomials, in *Operator Theory, Function Spaces, and Applications*, Oper. Theory Adv. Appl., vol. 255, Birkhäuser/Springer, Cham, 2016, pp. 123–136.

[3] Radomił Baran, Piotr Pikul, Hugo J. Woerdeman, and Michał Wojtylak: *Contractive realization theory for the annulus and other intersections of discs on the Riemann sphere.* preprint, arXiv:2504.03236 [math.FA], 2025.